

THE TRACIAL ROKHLIN PROPERTY FOR AUTOMORPHISMS ON NON-SIMPLE C^* -ALGEBRAS

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ABSTRACT. Let A be a unital AF-algebra (simple or non-simple) and let α be an automorphism of A . Suppose that α has certain Rokhlin property and A is α -simple. Suppose also that there is an integer $J \geq 1$ such that $\alpha_{*0}^J = \text{id}_{K_0(A)}$, we show that $A \rtimes_{\alpha} \mathbb{Z}$ has tracial rank zero.

1. INTRODUCTION

We introduce certain Rokhlin property for automorphisms on unital C^* -algebras. The Rokhlin property in ergodic theory was adopted to the context of von Neumann algebras by Connes [1]. It was adopted by Herman and Oseleanu [7] for UHF-algebras. Rørdam [13] and Kishimoto [5] introduced the Rokhlin property to a much more general context of C^* -algebras, then Osaka and Phillips studied integer group actions which satisfy certain type of Rokhlin property on some simple C^* -algebras [12]. More recently, Lin studied the Rokhlin property for automorphisms on simple C^* -algebras [10].

Phillips proposed that how to introduce appropriate Rokhlin property to non-simple C^* -algebras. In this paper we attempt to introduce certain Rokhlin property to non-simple C^* -algebras, when C^* -algebra is simple, this Rokhlin property is weaker than the Rokhlin property in [10, 12]. If an integer group action of a C^* -algebra has this Rokhlin property, we can conclude that its crossed product is in the C^* -algebra class of tracial rank zero. In particular, these algebras all belong to the class known currently to be classifiable by K-theoretic invariants in the sense of the Elliott classification program. We hope that this case will lead us to more interesting in the Rokhlin property to non-simple C^* -algebras.

The organization of the paper is as follows. In Section 1, we briefly recall the notion of C^* -algebras, then we introduce certain Rokhlin property and discuss some property of crossed product $A \rtimes_{\alpha} \mathbb{Z}$ when an automorphism α of a C^* -algebra A has the Rokhlin property. In Section 2, we show that

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if A is a unital AF-algebra, suppose that $\alpha \in \text{Aut}(A)$ has the tracial cyclic Rokhlin property and A is α -simple, suppose also that there is an integer $J \geq 1$ such that $\alpha_{*0}^J = \text{id}_{K_0(A)}$. Then $A \rtimes_\alpha \mathbb{Z}$ has tracial rank zero.

2. THE TRACIAL ROKHLIN PROPERTY

We will use the following convention:

- (1) Let A be a C^* -algebra, let $a \in A$ be a positive element and let $p \in A$ be a projection. We write $[p] \leq [a]$ if there is a projection $q \in \overline{aAa}$ and a partial isometry $v \in A$ such that $v^*v = p$ and $vv^* = q$.
- (2) Let A be a C^* -algebra. We denote by $\text{Aut}(A)$ the automorphism group of A . If A is unital and $u \in A$ is a unitary, we denote by adu the inner automorphism defined by $\text{adu}(a) = u^*au$ for all $a \in A$.
- (3) Let $x \in A, \varepsilon > 0$ and $\mathcal{F} \subset A$. We write $x \in_\varepsilon \mathcal{F}$, if $\text{dist}(x, \mathcal{F}) < \varepsilon$, or there is $y \in \mathcal{F}$ such that $\|x - y\| < \varepsilon$.
- (4) Let A be a C^* -algebra and $\alpha \in \text{Aut}(A)$. We say A is α -simple if A does not have any non-trivial α -invariant closed two-sided ideals.
- (5) A unital C^* -algebra is said to have real rank zero, written $\text{RR}(A) = 0$, if the set of invertible self-adjoint elements is dense in self-adjoint elements of A . Note that every unital AF-algebra has real rank zero.
- (6) A unital C^* -algebra A has the (SP)-property if every non-zero hereditary C^* -subalgebra of A has a non-zero projection. Note that every C^* -algebra A with real rank zero has the (SP)-property.
- (7) Let $T(A)$ be the tracial state space of a unital C^* -algebra A . It is a compact convex set.
- (8) we say the order on projection over a unital C^* -algebra A is determined by traces, if for any two projections $p, q \in A, \tau(p) < \tau(q)$ for all $\tau \in T(A)$ implies that p is equivalent to a projection $p' \leq q$.

Definition 2.1. We denote by $\mathcal{I}^{(0)}$ the class of all finite dimensional C^* -algebras, and denote by $\mathcal{I}^{(k)}$ the class of all unital C^* -algebras which are unital hereditary C^* -subalgebras of C^* -algebras of the form $C(X) \otimes F$, where X is a k -dimensional finite CW complex and $F \in \mathcal{I}^{(0)}$.

We recall the definition of tracial topological rank of C^* -algebras.

Definition 2.2. ([8]) Let A be a unital simple C^* -algebra. Then A is said to have tracial (topological) rank no more than k if for any $\varepsilon > 0$, any finite set $\mathcal{F} \subset A$, and any non-zero positive element $a \in A$, there exist a nonzero projection $p \in A$ and a C^* -subalgebra $B \in \mathcal{I}^{(k)}$ with $1_B = p$ such that:

- (1) $\|px - xp\| < \varepsilon$ for all $x \in \mathcal{F}$.
- (2) $pxp \in_\varepsilon B$ for all $x \in \mathcal{F}$.
- (3) $[1 - p] \leq [a]$.

If A has tracial rank no more than k , we will write $\text{TR}(A) \leq k$. If furthermore, $\text{TR}(A) \not\leq k - 1$, then we say $\text{TR}(A) = k$.

Definition 2.3. Let A be a unital C^* -algebra and let $\alpha \in \text{Aut}(A)$. Let $a \in A$ be a positive element and let $p \in A$ be a projection. We say $[p] \leq_\alpha [a]$ if there exist the mutually orthogonal projections p_i , the mutually orthogonal positive elements a_i and $s_i \in \mathbb{Z}$ for $i = 1, 2, \dots, n$ such that $p = \sum_{i=1}^n p_i$, $\{a_i\}_{i=1}^n$ belong to the hereditary C^* -subalgebra generated by a , and $[\alpha^{s_i}(p_i)] \leq [a_i]$, $i = 1 \dots n$.

By this definition, we can compare non-zero positive elements with full positive elements by the action of α .

Example 2.4. Let $A = A_0 \oplus A_0$, where A_0 is an infinite dimensional unital simple C^* -algebra with real rank zero, let $\alpha \in \text{Aut}(A)$ such that $\alpha(a_0, b_0) = (b_0, a_0)$, where $a_0, b_0 \in A_0$, then for any non-zero projection $q \in A$, there exists a projection $p = (p_1, p_2) \in A$, $p_1 \neq 0, p_2 \neq 0$ such that $[p] \leq_\alpha [q]$.

Definition 2.5. Let A be a unital C^* -algebra and let $\alpha \in \text{Aut}(A)$. We say α has the tracial Rokhlin property if for every $\varepsilon > 0$, every $n \in \mathbb{N}$, every nonzero positive element $a \in A$, every finite set $\mathcal{F} \subset A$, $\mathcal{F} = \{p_1, \dots, p_m, a_1, \dots, a_s\}$, where $\{p_i\}, i = 1, \dots, m$ are the mutually orthogonal projections, there are the mutually orthogonal projections $e_1, e_2, \dots, e_n \in A$ such that:

- (1) $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $1 \leq j \leq n-1$.
- (2) $\|e_j b - b e_j\| < \varepsilon$ for $1 \leq j \leq n$ and all $b \in \mathcal{F}$.
- (3) $\|e_1 p_j e_1\| \geq 1 - \varepsilon$ for $1 \leq j \leq m$.
- (4) With $e = \sum_{j=1}^n e_j$, $[1 - e] \leq_\alpha [a]$.

When A is a unital simple C^* -algebra, above definition is weaker than the Rokhlin property in [10, 12]. We weak the condition (4) to require positive elements can be compared by action of α .

We define a slightly stronger version of the tracial Rokhlin property.

Definition 2.6. Let A be a unital C^* -algebra and let $\alpha \in \text{Aut}(A)$. We say α has the tracial cyclic Rokhlin property if for every $\varepsilon > 0$, every $n \in \mathbb{N}$, every nonzero positive element $a \in A$, every finite set $\mathcal{F} \subset A$, $\mathcal{F} = \{p_1, \dots, p_m, a_1, \dots, a_s\}$, where $\{p_i\}, i = 1, \dots, m$ are the mutually orthogonal projections, there are the mutually orthogonal projections $e_1, e_2, \dots, e_n \in A$ such that:

- (1) $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $1 \leq j \leq n$, where $e_{n+1} = e_1$.
- (2) $\|e_j b - b e_j\| < \varepsilon$ for $1 \leq j \leq n$ and all $b \in \mathcal{F}$.
- (3) $\|e_1 p_j e_1\| \geq 1 - \varepsilon$ for $1 \leq j \leq m$.
- (4) With $e = \sum_{j=1}^n e_j$, $[1 - e] \leq_\alpha [a]$.

The only difference between the tracial Rokhlin property and the tracial cyclic Rokhlin property is that in condition (1), we require that $\|\alpha(e_n) - e_1\| < \varepsilon$.

Theorem 2.7. Let A be a unital C^* -algebra with real rank zero, let $\alpha \in \text{Aut}(A)$ have the tracial Rokhlin property. Then A is α -simple if and only if the crossed product $A \rtimes_\alpha \mathbb{Z}$ is simple.

Proof. Let I be an α -invariant norm closed two-sided ideal of A . Then $I \rtimes_\alpha \mathbb{Z}$ is a norm closed two-sided ideal of $A \rtimes_\alpha \mathbb{Z}$ by Lemma 1 of [3].

Conversely, let a be a positive element of the C^* -algebra A , $\mathcal{F} = \{a_i; i = 1, 2, \dots, n\}$ elements of A , $s_i \in \mathbb{N}, i = 1, 2, \dots, n$ and $\varepsilon > 0$, we prove that there exists a positive element $x \in A$ with $\|x\| = 1$ such that

$$\|xax\| \geq \|a\| - \varepsilon, \quad \|xa_i\alpha^{s_i}(x)\| \leq \varepsilon, \quad \|xa_i - a_ix\| < \varepsilon, i = 1, 2, \dots, n. \quad (*)$$

Because A has real rank zero, let $\varepsilon > 0$, by Theorem 3.2.5 of [9], there are mutually orthogonal projections p_1, p_2, \dots, p_m and positive real numbers $\lambda_1, \lambda_2, \dots, \lambda_m$ such that $\|a - \sum_{i=1}^m \lambda_i p_i\| < \varepsilon/3$, let $a_0 = \sum_{i=1}^m \lambda_i p_i$, $C = \max\{\|a_i\|; i = 1, \dots, n\}$ and $N = \max\{s_1, s_2, \dots, s_n\}$.

Let $\varepsilon_0 = \min\{\frac{\varepsilon}{3\|a_0\|}, \frac{\varepsilon}{(N+2)C}\}$.

Apply the tracial Rokhlin property with N in place of n , with ε_0 in place of ε . We can obtain e_1, e_2, \dots, e_N , such that

- (1) $\|\alpha(e_j) - e_{j+1}\| < \varepsilon_0$ for $1 \leq j \leq N-1$,
- (2) $\|e_j a_i - a_i e_j\| < \varepsilon_0$ for $1 \leq j \leq N$ and $1 \leq i \leq n$,
- (3) $\|e_1 p_j e_1\| \geq 1 - \varepsilon_0$ for $1 \leq j \leq m$,

then

$$\|e_1 a_0 e_1\| = \|\sum_{i=1}^m \lambda_i e_1 p_i e_1\| \geq \|\lambda_i e_1 p_i e_1\| \geq \lambda_i (1 - \varepsilon_0), i = 1, 2, \dots, m.$$

we get

$$\|e_1 a_0 e_1\| \geq \|a_0\| (1 - \varepsilon_0) \geq \|a_0\| - \frac{\varepsilon}{3}.$$

then

$$\begin{aligned} \|e_1 a e_1\| &= \|e_1 a_0 e_1 + e_1 a e_1 - e_1 a_0 e_1\| \geq \|e_1 a_0 e_1\| - \|e_1 a e_1 - e_1 a_0 e_1\| \\ &\geq \|e_1 a_0 e_1\| - \frac{\varepsilon}{3} \geq \|a_0\| - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} \geq \|a\| - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} = \|a\| - \varepsilon. \end{aligned}$$

$$\begin{aligned} &\|e_1 a_i \alpha^{s_i}(e_1)\| \\ &= \|e_1 a_i \alpha^{s_i}(e_1) - e_1 a_i \alpha^{s_i-1}(e_1) + e_1 a_i \alpha^{s_i-1}(e_1) + \dots + e_1 a_i \alpha^1(e_1)\| \\ &< \|e_1 a_i \alpha^1(e_1)\| + (s_i - 1)\varepsilon_0 \|a_i\| < \|a_i e_1 \alpha^1(e_1)\| + s_i \varepsilon_0 \|a_i\| \\ &< (s_i + 1)\varepsilon_0 \|a_i\| < \varepsilon. \end{aligned}$$

So we get (*). Apply this condition and A is α -simple, we can complete the proof as same as Theorem 3.1 of [4], we omit it. \square

Apply the (*) and the same proof of Theorem 4.2 of [2], we can get the following result.

Theorem 2.8. *Let A be a unital C^* -algebra with real rank zero and let $\alpha \in \text{Aut}(A)$ have the tracial Rokhlin property and A is α -simple. Then any non-zero hereditary C^* -subalgebra of the crossed product $A \rtimes_{\alpha} \mathbb{Z}$ has a non-zero projection which is equivalent to a projection in A .*

Lemma 2.9. *Let $B = M_{r(1)} \oplus M_{r(2)} \oplus \dots \oplus M_{r(l)}$ be a finite dimensional C^* -subalgebra of a unital C^* -algebra A , Let $e_{i,j}^{(s)} \in B$ be a system of matrix units for $M_{r(s)}$, $s = 1, 2, \dots, l$. Then for any $\delta > 0$, there exists $\sigma > 0$ satisfying the following: If $\|pe_{i,i}^{(s)} - e_{i,i}^{(s)}p\| < \sigma$ and $\|pe_{i,i}^{(s)}p\| > 1/2$ for $s =$*

$1, 2, \dots, l, i = 1, 2, \dots, r(s)$, then there is an monomorphism $\varphi : B \rightarrow pAp$ such that $\|pbp - \varphi(b)\| < \delta\|b\|$ for all $b \in B$.

Proof. It follows from the argument in section 2.5 of [9] and Proposition 2.3 of [11]. \square

Proposition 2.10. *Let A be a unital C^* -algebra, Suppose that $\alpha \in \text{Aut}(A)$ is approximately inner and has the tracial Rokhlin property, if for any closed two-sided ideal I of C^* -algebra A , there is $n \in \mathbb{N}$, n only depends on I , such that $K_0(A/I)$ is not n -divisible, then A is α -simple.*

Proof. Suppose that A is not α -simple, so there exists a closed two-sided ideal I of C^* -algebra A such that $\alpha(I) = I$. By the hypothesis, there is $n \in \mathbb{N}$ such that $K_0(A/I)$ is not n -divisible.

Let $a \in I$ be a non-zero positive element, and $0 < \varepsilon < 1$, there are the mutually orthogonal projections $e_1, e_2, \dots, e_n \in A$ such that

- (1) $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $1 \leq j \leq n-1$,
- (2) With $e = \sum_{j=1}^n e_j$, $[1-e] \leq_\alpha [a]$.

Because α is approximately inner and (1), we have $[e_1] = [e_2] = \dots = [e_n]$ in $K_0(A)$.

If $p \in A$ is a projection such that $[p] \leq [b]$, where $b \in I$ is a positive element, then there is a $v \in A$ such that $v^*v = p$ and $vv^* \in \overline{bAb} \subset I$, if $\pi : A \rightarrow A/I$ denotes quotient map, $\pi(v)\pi(v^*) = 0$ in A/I , $\pi(v) = 0$ in A/I , then $p \in I$.

In (2), $[1-e] \leq_\alpha [a]$, by the definition of \leq_α , $a \in I$ and the discuss above, we have $1-e \in I$, so $\pi(1-e) = 0$, $[1-e] = 0$ in $K_0(A/I)$, then $n[e_1] = [1]$ in $K_0(A/I)$, this is contradictory to $K_0(A/I)$ is not n -divisible. \square

3. MAIN RESULT

In the proof of Theorem 3.3, we first prove $TR(A \rtimes_\alpha \mathbb{Z}) \leq 1$, then use the following Lemma 3.1 to prove $RR(A \rtimes_\alpha \mathbb{Z}) = 0$. The following Lemma is similar to Lemma 2.5 of [12].

Lemma 3.1. *Let A be a unital C^* -algebra with real rank zero and let $\alpha \in \text{Aut}(A)$ have the tracial Rokhlin property. Suppose that A is α -simple and the order on projection over $A \rtimes_\alpha \mathbb{Z}$ is determined by traces. Let $\iota : A \rightarrow A \rtimes_\alpha \mathbb{Z}$ be the inclusion map. Then for every finite set $F \subset A \rtimes_\alpha \mathbb{Z}$, every $\varepsilon > 0$, every nonzero positive element $z \in A \rtimes_\alpha \mathbb{Z}$, and every sufficiently large $n \in \mathbb{N}$ (depending on F, ε and z), there exist a projection $e \in A \subset A \rtimes_\alpha \mathbb{Z}$, a unital subalgebra $D \subset e(A \rtimes_\alpha \mathbb{Z})e$, a projection $p \in D$, a projection $f \in A$, and an isomorphism $\varphi : M_n \otimes fAf \rightarrow D$, such that:*

- (1) *With $(e_{j,k})$ being the standard system of matrix units for M_n , we have $\varphi(e_{1,1} \otimes a) = \iota(a)$ for all $a \in fAf$ and $\varphi(e_{k,k} \otimes 1) \in \iota(A)$ for $1 \leq k \leq n$.*
- (2) *With $(e_{j,k})$ as in (1), we have $\|\varphi(e_{j,j} \otimes a) - \alpha^{j-1}(\iota(a))\| \leq \varepsilon\|a\|$ for all $a \in fAf$.*

(3) For every $a \in F$, there exists $b_1, b_2 \in D$ such that $\|pa - b_1\| < \varepsilon$, $\|ap - b_2\| < \varepsilon$, and $\|b_1\|, \|b_2\| \leq \|a\|$.

(4) There is $m \in \mathbb{N}$ such that $2m/n < \varepsilon$ and $p = \sum_{j=m+1}^{n-m} \varphi(e_{j,j} \otimes 1)$.

(5) The projection $1 - p$ is Murray-von Neumann equivalent in $A \rtimes_\alpha \mathbb{Z}$ to a projection in the hereditary subalgebra of $A \rtimes_\alpha \mathbb{Z}$ generated by z and $\tau(1 - p) < \varepsilon$ for all $\tau \in T(A \rtimes_\alpha \mathbb{Z})$.

Proof. Let $\varepsilon > 0$, and let $F \subset A \rtimes_\alpha \mathbb{Z}$ be a finite set. Let $z \in A \rtimes_\alpha \mathbb{Z}$ be a nonzero positive element.

Let u be the standard unitary in the crossed product $A \rtimes_\alpha \mathbb{Z}$. We regard A as a subalgebra of $A \rtimes_\alpha \mathbb{Z}$ in the usual way. Choose $m \in \mathbb{N}$ such that for every $x \in F$ there are $a_l \in A$ for $-m \leq l \leq m$ such that $\|x - \sum_{l=-m}^m a_l u^l\| < \frac{\varepsilon}{2}$. For each $x \in F$ choose one such expression, and let $S \subset A$ be a finite set which contains all the coefficients used for all elements of F . Let $M = 1 + \sup_{a \in S} \|a\|$.

Since $A \rtimes_\alpha \mathbb{Z}$ has (SP)-property and is simple by Theorem 2.8 and Theorem 2.7, we can apply Lemma 3.5.7 of [9] to find nonzero orthogonal Murray-von Neumann equivalent projections $g_0, g_1, \dots, g_{2m} \in z(A \rtimes_\alpha \mathbb{Z})z$.

Since $A \rtimes_\alpha \mathbb{Z}$ is simple, g_0 is a nonzero projection, and the tracial state space $T(A \rtimes_\alpha \mathbb{Z})$ of $A \rtimes_\alpha \mathbb{Z}$ is weak-* compact, we have

$$\delta = \inf_{\tau \in T(A \rtimes_\alpha \mathbb{Z})} \tau(g_0) > 0.$$

Now let $n \in \mathbb{N}$ be any integer such that $n > \max(\frac{1}{\delta}, (N+2)(2m+1), \frac{4m}{\varepsilon})$.

Set $\varepsilon_0 = \frac{\varepsilon}{10(2m+1)n^2M}$.

Choose $\varepsilon_1 > 0$ so small that whenever e_1, e_2, \dots, e_n are mutually orthogonal projections in a unital C^* -algebra B and $u \in B$ is a unitary such that $\|ue_j u^* - e_{j+1}\| < \varepsilon_1$ for $1 \leq j \leq n$, then there is a unitary $v \in B$ such that $\|v - u\| < \varepsilon_0$ and $ve_j v^* = e_{j+1}$ for $1 \leq j \leq n$. We can apply Lemma 3.5.7 of [9] to find nonzero orthogonal Murray-von Neumann equivalent projections $h_1, h_2, \dots, h_{n+2} \in \overline{g_0(A \rtimes_\alpha \mathbb{Z})g_0}$ which are Murray-von Neumann equivalent in $A \rtimes_\alpha \mathbb{Z}$. Further apply Theorem 2.8 to find a nonzero projection $q \in A$ which is Murray-von Neumann equivalent in $A \rtimes_\alpha \mathbb{Z}$ to a projection in $\overline{h_1(A \rtimes_\alpha \mathbb{Z})h_1}$.

Apply the tracial Rokhlin property with $n-1$ in place of n , with S in place of F , with $\min(1, \varepsilon_0, \varepsilon_1)$ in place of ε , and with q in place of x . Call the resulting projections e_1, e_2, \dots, e_n , and let $e = \sum_{j=1}^n e_j$, $[1-e] \leq_\alpha [q]$. Apply the choice of ε_1 to these projections and the standard unitary u , obtaining a unitary $v \in A \rtimes_\alpha \mathbb{Z}$ as in the previous paragraph.

We can get the conditions (1),(2),(3),(4) by the same proof of Lemma 2.5 of [12]. We omit them.

It remains to verify Condition (5) of the conclusion. We have

$$1 - p = 1 - e + \sum_{j=1}^m e_j + \sum_{j=n-m+1}^n e_j.$$

By construction we have $[1 - e] \leq_\alpha [h_1] \leq [g_0]$. Now let τ be any tracial state on $A \rtimes_\alpha \mathbb{Z}$. Then $\tau(e_j) = \tau(e_1)$ for all j , whence $\tau(e_j) \leq \frac{1}{n}$. The inequality $n > \frac{1}{\delta} \geq \frac{1}{\tau(g_0)}$ therefore implies $\tau(e_j) < \tau(g_0)$. Since all g_j are Murray-von Neumann equivalent, it follows that for any tracial state τ on $A \rtimes_\alpha \mathbb{Z}$, we have $\tau(e_j) < \tau(g_j)$ and $\tau(e_{n-j}) < \tau(g_{m+j})$ for $1 \leq j \leq m$. So the order on projection over $A \rtimes_\alpha \mathbb{Z}$ is determined by traces implies that $e_j \leq g_j$ and $e_{n-j} \leq g_{m+j}$ in $A \rtimes_\alpha \mathbb{Z}$ for $1 \leq j \leq m$. Thus $[1 - p] \leq_\alpha [\sum_{j=0}^{2m} g_j]$ which is a projection in the hereditary subalgebra $\overline{z(A \rtimes_\alpha \mathbb{Z})z}$.

$$\tau(1 - p) = \tau(1 - e) + \tau\left(\sum_{j=1}^m e_j + \sum_{j=n-m+1}^n e_j\right) \leq \frac{1}{2m(n+2)} + \frac{2m}{n} < \varepsilon.$$

This is Condition (5) of the conclusion. \square

Theorem 3.2. *Let A be a unital C^* -algebra with real rank zero and let $\alpha \in \text{Aut}(A)$ have the tracial Rokhlin property. Suppose that A is α -simple and the order on projection over $A \rtimes_\alpha \mathbb{Z}$ is determined by traces. Then $A \rtimes_\alpha \mathbb{Z}$ has real rank zero.*

Proof. By applying Lemma 3.1 and the same proof of Theorem 4.5 of [12]. \square

Theorem 3.3. *Let A be a unital AF-algebra, Suppose that $\alpha \in \text{Aut}(A)$ has the tracial cyclic Rokhlin property and A is α -simple. Suppose also that there is an integer $J \geq 1$ such that $\alpha_{*0}^J = \text{id}_{K_0(A)}$. Then $\text{TR}(A \rtimes_\alpha \mathbb{Z}) = 0$.*

Proof. By Theorem 2.7, $A \rtimes_\alpha \mathbb{Z}$ is a unital simple C^* -algebra.

Let $0 < \varepsilon < 1$ and $\mathcal{F} \subset A \rtimes_\alpha \mathbb{Z}$ be a finite set. To simplify notation, without loss of generality, we may assume that $\mathcal{F} = \mathcal{F}_0 \cup \{u\}$, where $\mathcal{F}_0 \subset A$ is a finite subset of the unit ball which contains 1_A and u is a unitary which implements α , i.e., $\alpha(a) = u^* a u$ for all $a \in A$. Choose an integer k which is a multiple of J such that $2\pi/(k-2) < \varepsilon/16$. Put $\mathcal{F}_1 = \mathcal{F}_0 \cup \{u^i a (u^*)^i : a \in \mathcal{F}_0, -k \leq i \leq k\}$.

Fix $b_0 \in (A \rtimes \mathbb{Z})_+ \setminus \{0\}$. It follows from Theorem 2.8 that there is a nonzero projection $r_0 \in A$ which is equivalent to a nonzero projection in the hereditary C^* -subalgebra generated by b_0 .

Let $\delta = \varepsilon/16k^2$. Since A is a unital AF-algebra, denoted by $A = \overline{\bigcup_{m=1}^\infty A_m}$, where A_m is a finite-dimensional C^* -algebra for $m = 1, 2, \dots$, then there is a large enough $m \in \mathbb{N}$ such that $b \in_\delta A_m$ for all $b \in \mathcal{F}_1$ and $1_A \in A_m$. Let $A_m = M_{r(1)} \oplus M_{r(2)} \oplus \dots \oplus M_{r(l)}$. Note $[(u^k)^* e u^k] = [e]$ in $K_0(A)$ for all projection $e \in A_m$. By Theorem 3.4.6 of [9], there exists a unitary $w \in U(A)$ such that $w^*(u^k)^* b u^k w = b$ for all $b \in A_m$. Because A is a AF-algebra, $w \in U_0(A)$. By Lemma 2.6 of [10], we have the unitaries $w_i, i = 1, 2, \dots, k-1$ associated with finite dimensional C^* -subalgebra A_m such that $w = w_1 w_2 \dots w_{k-1}$, $\|w_i - 1\| \leq \pi/(k-2)$. Let \mathcal{G}_0 be a finite subset of A_m which, for each $b \in \mathcal{F}_1$ contains an element $a(b)$ such that $\|a(b) - b\| < \delta$, contains a systems of matrix units for each simple summand of A_m .

Define $\mathcal{F}_2 = \{u^i b u^{-i} : b \in \mathcal{G}_0, -k \leq i \leq k\}$ and let $w_k = 1$

$\mathcal{F}_3 = \{(w_{i_1} w_{i_1+1} \cdots w_i) a (w_{i_2} w_{i_2+1} \cdots w_i)^* : a \in \mathcal{F}_1 \cup \mathcal{F}_2, 1 \leq i, i_1, i_2 \leq k, i_1 \leq i, i_2 \leq i\}$. Note that $w, w_i \in \mathcal{F}_3, i = 1, 2, \dots, k-1$.

Since α has the tracial cyclic Rokhlin property, let $e_{i,j}^{(s)} \in A_m$ be a system of matrix units for $M_{r(s)}, s = 1, 2, \dots, l$. Let $\sigma > 0$ be associated with A_m and δ in Lemma 2.9. Let $\eta = \min\{\delta, \sigma\}$, there exist projections $e_1, e_2, \dots, e_k \in A$ such that:

- (1) $\|\alpha(e_i) - e_{i+1}\| < \eta/k$ for $1 \leq i \leq k, e_{k+1} = e_1$.
- (2) $\|e_i a - a e_i\| < \eta/k$ for $a \in \mathcal{F}_3$.
- (3) $\|e_1 e_{jj}^{(s)} e_1\| \geq 1 - \eta/k$ for $s = 1, 2, \dots, l, j = 1, 2, \dots, r(s)$.
- (4) $[1 - \sum_{i=1}^k e_i] \leq_\alpha [r_0]$.

Set $p = \sum_{i=1}^k e_i$. From (1) above, one estimates that

$$\begin{aligned} \|up - pu\| &= \left\| \sum_{i=1}^k u e_{i+1} - \sum_{i=1}^k e_i u \right\| \leq \sum_{i=1}^k \|u e_{i+1} - e_i u\| \\ &= \sum_{i=1}^k \|u e_{i+1} - u \alpha(e_i)\| < \eta. \end{aligned}$$

By (1) above, one sees that there is a unitary $v \in A$ such that $\|v - 1\| < 2\eta/k$ and $v^* u^* e_i u v = e_{i+1}, i = 1, 2, \dots, k$. Set $u_1 = uv$. Then $u_1^* e_i u_1 = e_{i+1}, i = 1, 2, \dots, k$ and $e_{k+1} = e_1$. In particular, $u_1^k e_1 = e_1 u_1^k$. For any $a \in \mathcal{F}_3 \cap A_m$ (since $w \in \mathcal{F}_3$), $e_1 w^* e_1 (u_1^k)^* e_1 a e_1 u_1^k e_1 w e_1 \approx_{3\eta/k} e_1 a e_1$. By (2), (3) above, it then follows from Lemma 2.9, there is a monomorphism $\varphi : A_m \rightarrow e_1 A e_1$ such that $\|\varphi(a) - e_1 a e_1\| < \delta \|a\|$ for all $a \in A_m$.

By applying Lemma 2.9 of [10], we obtain unitaries $x, x_1, x_2, \dots, x_{k-1} \in U_0(e_1 A e_1)$ such that $\|x - e_1 w e_1\| < \delta, \|x_i - e_1 w_i e_1\| < \delta, x = x_1 x_2 \cdots x_{k-1}$ and $x^* (u_1^k)^* a u_1^k x = a$ for all $a \in \varphi(A_m)$.

Let $Z = \sum_{i=1}^k e_i u_1^{k+1-i} x_i (u_1^{k-i})^* + (1-p)u_1$. Define $B = \varphi(A_m)$, then

$$\begin{aligned} \|Z - u_1\| &\leq \max_i \{\|x_i - e_1\|\} \leq \max_i \{\|x_i - e_1 w_i e_1\| + \|e_1 w_i e_1 - e_1\|\} \\ &< \delta + \eta/k + \pi/(k-2), \end{aligned}$$

$(Z^k)^* b Z^k = b$ for all $b \in B$ and

$$(Z^i)^* e_1 Z^i \leq e_{i+1}, Z^i = u_1^k (x_1 x_2 \cdots x_i) (u_1^{k-i})^*, i = 1, 2, \dots, k (e_{k+1} = e_1).$$

Write $B = C_1 \oplus C_2 \oplus \cdots \oplus C_N$ and $\{c_{is}^{(j)}\}$ be the matrix units for $C_j, j = 1, 2, \dots, N$, where $C_j = M_{R(j)}$ and put $q = 1_B$.

Define $D_0 = B \oplus \bigoplus_{i=1}^{k-1} Z^{i*} B Z^i$, and D_1 the C^* -subalgebra generated by B and $c_{ss}^{(j)} Z^i, s = 1, 2, \dots, R(j), j = 1, 2, \dots, N$ and $i = 0, 1, 2, \dots, k-1$. Then $D_1 \cong B \otimes M_k$ and $D_1 \supset D_0$.

Define $q_{ss}^{(j)} = \sum_{i=0}^{k-1} Z^i {}^* c_{ss}^{(j)} Z^i$, $q^{(j)} = \sum_{s=1}^{R(j)} q_{ss}^{(j)}$ and $Q = \sum_{j=1}^N q^{(j)} = 1_{D_1}$. Note that $Q = \sum_{i=0}^{k-1} (Z^i)^* q Z^i$. Note that

$$\begin{aligned} q_{ss}^{(j)} Z &= \left(\sum_{i=0}^{k-1} Z^i {}^* c_{ss}^{(j)} Z^i \right) Z = Z \sum_{i=0}^{k-1} (Z^{i+1})^* c_{ss}^{(j)} Z^{i+1} \\ &= Z \left(\sum_{i=1}^{k-1} Z^i {}^* c_{ss}^{(j)} Z^i + c_{ss}^{(j)} \right) = Z q_{ss}^{(j)}. \end{aligned}$$

It follows from Lemma 2.11 of [10] that $c_{11}^{(j)}, c_{11}^{(j)} Z^i$ and $c_{11}^{(j)} Z^k c_{11}^{(j)}$ generate a C^* -subalgebra which is isomorphic to $C(X_j) \otimes M_k$ for some compact subset $X_j \subset S^1$. Moreover, $q_{ss}^{(j)} Z q_{ss}^{(j)}$ is in the C^* -subalgebra. Let D be the C^* -subalgebra generated by D_1 and $c_{11}^{(j)} Z^k c_{11}^{(j)}$. Then $D \cong \oplus_{j=1}^N C(X_j) \otimes B \otimes M_k$. It follows that $q^{(j)}$ and Q commutes with Z . Therefore $QZQ \in D$. Thus,

$$\begin{aligned} \|Qu - uQ\| &\leq \|Qu - Qu_1\| + \|Qu_1 - QZ\| + \|ZQ - u_1Q\| + \|u_1Q - uQ\| \\ &< 4\eta/k + 2\delta + 2\pi/(k-2) < \varepsilon. \end{aligned}$$

From $QZQ \in D$, we also have $QuQ \in_\varepsilon D$.

For $b \in \mathcal{F}_0$, we compute that

$$\begin{aligned} (Z^i)^* q (Z^i) b &= (Z^i)^* q u_1^k (x_1 x_2 \cdots x_i) (u_1^{k-i})^* b \\ &\approx {}_{k\delta+2\eta} (Z^i)^* q u_1^k (w_1 w_2 \cdots w_i) (u_1^{k-i})^* b \\ &= (Z^i)^* q u_1^k (w_1 w_2 \cdots w_i) (u_1^{k-i})^* b u_1^{k-i} (w_1 w_2 \cdots w_i)^* (u_1^k)^* [u_1^{k-i} (w_1 w_2 \cdots w_i)^* (u_1^k)^*]^* \end{aligned}$$

Put $c_i = (u_1^{k-i})^* b u_1^{k-i}$, then $c_i \in \mathcal{F}_1$. There is $a_i \in \mathcal{G}_0 \subset A_m$ such that $\|c_i - a_i\| < \delta$.

Since $(w_1 w_2 \cdots w_i) \mathcal{F}_1 (w_1 w_2 \cdots w_i)^* \subset \mathcal{F}_3$, then

$$\begin{aligned} &(Z^i)^* q u_1^k (w_1 w_2 \cdots w_i) (u_1^{k-i})^* b u_1^{k-i} (w_1 w_2 \cdots w_i)^* (u_1^k)^* [u_1^{k-i} (w_1 w_2 \cdots w_i)^* (u_1^k)^*]^* \\ &= (Z^i)^* q u_1^k (w_1 w_2 \cdots w_i) c_i (w_1 w_2 \cdots w_i)^* (u_1^k)^* [u_1^{k-i} (w_1 w_2 \cdots w_i)^* (u_1^k)^*]^* \\ &\approx {}_\delta (Z^i)^* q u_1^k (w_1 w_2 \cdots w_i) a_i (w_1 w_2 \cdots w_i)^* (u_1^k)^* [u_1^{k-i} (w_1 w_2 \cdots w_i)^* (u_1^k)^*]^* \\ &\approx {}_\delta (Z^i)^* e_1 u_1^k (w_1 w_2 \cdots w_i) a_i (w_1 w_2 \cdots w_i)^* (u_1^k)^* [u_1^{k-i} (w_1 w_2 \cdots w_i)^* (u_1^k)^*]^* \\ &\approx {}_{\eta/k} (Z^i)^* u_1^k (w_1 w_2 \cdots w_i) a_i (w_1 w_2 \cdots w_i)^* (u_1^k)^* e_1 [u_1^{k-i} (w_1 w_2 \cdots w_i)^* (u_1^k)^*]^* \\ &\approx {}_\delta (Z^i)^* u_1^k (w_1 w_2 \cdots w_i) c_i (w_1 w_2 \cdots w_i)^* (u_1^k)^* q [u_1^{k-i} (w_1 w_2 \cdots w_i)^* (u_1^k)^*]^* \\ &\approx {}_\delta (Z^i)^* u_1^k (w_1 w_2 \cdots w_i) (u_1^{k-i})^* b u_1^{k-i} (w_1 w_2 \cdots w_i)^* (u_1^k)^* q [u_1^{k-i} (w_1 w_2 \cdots w_i)^* (u_1^k)^*]^* \\ &\approx {}_{k\delta+2\eta} b (Z^i)^* q Z^i. \end{aligned}$$

Hence

$$\|(Z^i)^* q Z^i b - b (Z^i)^* q Z^i\| < 2(k\delta + 2\eta + \delta + \delta) + \eta/k < \varepsilon/k, k = 0, 1, \dots, k-1.$$

Therefore, for $b \in \mathcal{F}_0$, $\|Qb - bQ\| < k \cdot (\varepsilon/k) = \varepsilon$.

It follows that $\|Qa - aQ\| < \varepsilon$ for all $a \in \mathcal{F}$.

For any $b \in \mathcal{F}_0$, a same estimation above shows that

$$\|qZ^ib(Z^i)^*q - qu_1^k(w_1w_2 \cdots w_i)(u^{k-i})^*bu^{k-i}(w_1w_2 \cdots w_i)^*(u_1^k)^*q\| < 2k\delta + 4\eta$$

However, $qu_1^k(w_1w_2 \cdots w_i)(u^{k-i})^*bu^{k-i}(w_1w_2 \cdots w_i)^*(u_1^k)^*q \in_{\delta+2\delta+4\eta/k} B$.

It follows that, for $b \in \mathcal{F}_0$,

$$(Z^i)^*qZ^ib(Z^i)^*qZ^i \in_{\varepsilon/k} (Z^i)^*BZ^i, i = 0, 1, \dots, k-1.$$

we obtain that $QbQ \in_{\varepsilon} D_1 \subset D$.

we obtain that $QaQ \in_{\varepsilon} D$ for all $a \in \mathcal{F}$.

Because $[1 - \sum_{i=1}^k e_i] = [1 - p] \leq_{\alpha} [r_0]$ in A , there exist the mutually orthogonal projections p_i , the mutually orthogonal positive elements a_i and $s_i \in \mathbb{Z}$ for $i = 1, 2, \dots, n$ such that $p = \sum_{i=1}^n p_i$, $\{a_i\}_{i=1}^n$ belong to the hereditary C^* -subalgebra generated by r_0 , and $[\alpha^{s_i}(p_i)] \leq [a_i]$, $i = 1 \cdots n$. Because $[\alpha^{s_i}(p_i)] = [u^{s_i}p_i(u^{s_i})^*] = [p_i]$ in $A \rtimes_{\alpha} \mathbb{Z}$. we obtain that $[1 - \sum_{i=1}^k e_i] \leq [r_0]$ in $A \rtimes_{\alpha} \mathbb{Z}$.

We can compute that

$$[1 - Q] \leq [1 - \sum_{i=1}^k e_i] \leq [r_0] \leq [b_0].$$

So $\text{TR}(A \rtimes_{\alpha} \mathbb{Z}) \leq 1$.

The order on projection over $A \rtimes_{\alpha} \mathbb{Z}$ is determined by traces by Theorem 3.7.2 of [9].

By applying Theorem 3.2, we have $\text{RR}(A \rtimes_{\alpha} \mathbb{Z}) = 0$. By Lemma 3.2 of [10], we conclude that $\text{TR}(A \rtimes_{\alpha} \mathbb{Z}) = 0$. \square

Corollary 3.4. *Let A be a unital AF-algebra, Suppose that $\alpha \in \text{Aut}(A)$ has the tracial cyclic Rokhlin property and A is α -simple. Suppose also that there is an integer $J \geq 1$ such that $\alpha_{*0}^J = \text{id}_{K_0(A)}$. Then the restriction map is a bijection from the tracial states of $A \rtimes_{\alpha} \mathbb{Z}$ to the α -invariant tracial states of A .*

Proof. Since A has real rank zero and $A \rtimes_{\alpha} \mathbb{Z}$ also has real rank zero by Theorem 3.3, this follows from Proposition 2.2 of [6]. \square

Example 3.5. *Let $A = A_0 \oplus A_0$, where A_0 is an infinite dimensional unital simple AF-algebra. Let $\beta \in \text{Aut}(A_0)$ be an approximately inner automorphism of A_0 and have the traical cyclic Rokhlin property in [10]. Define $\alpha \in \text{Aut}(A)$ by $\alpha(a, b) = (\beta(b), \beta(a))$, then $\text{TR}(A \rtimes_{\alpha} \mathbb{Z}) = 0$.*

Obviously, A is α -simple. Because β is an approximately inner automorphism of A_0 , therefore $\beta_{*0} = \text{id}_{K_0(A_0)}$, then we have $\alpha_{*0}^2 = \text{id}_{K_0(A)}$.

Because β is an approximately inner automorphism of A_0 and has the traical cyclic Rokhlin property in [10], furthermore by applying Lemma 2.8 of [10], it is easy to verify that α has the traical cyclic Rokhlin property in this paper.

So (A, α) satisfies the conditions of Theorem 3.3, then we have $\text{TR}(A \rtimes_{\alpha} \mathbb{Z}) = 0$.

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